

Serre Dimension of Monoid Algebras

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Abstract

Let R be a commutative Noetherian ring of dimension d , M a commutative cancellative torsion-free monoid of rank r and P a finitely generated projective $R[M]$ -module of rank t .

(1) Assume M is Φ -simplicial seminormal. (i) If $M \in \mathcal{C}(\Phi)$, then $\text{Serre dim } R[M] \leq d$. (ii) If $r \leq 3$, then $\text{Serre dim } R[\text{int}(M)] \leq d$.

(2) If $M \subset \mathbb{Z}_+^2$ is a normal monoid of rank 2, then $\text{Serre dim } R[M] \leq d$.

(3) Assume M is c -divisible, $d = 1$ and $t \geq 3$. Then $P \cong \wedge^t P \oplus R[M]^{t-1}$.

(4) Assume R is a uni-branched affine algebra over an algebraically closed field and $d = 1$. Then $P \cong \wedge^t P \oplus R[M]^{t-1}$.

1 Introduction

Throughout rings are commutative Noetherian with 1; projective modules are finitely generated and of constant rank; monoids are commutative cancellative torsion-free; \mathbb{Z}_+ denote the additive monoid of non-negative integers.

Let A be a ring and P a projective A -module. An element $p \in P$ is called *unimodular*, if there exists $\phi \in \text{Hom}(P, A)$ such that $\phi(p) = 1$. We say Serre dimension of A (denoted as $\text{Serre dim } A$) is $\leq t$, if every projective A -module of rank $\geq t + 1$ has a unimodular element. Serre dimension of A measures the surjective stabilization of the Grothendieck group $K_0(A)$. Serre's problem on the freeness of projective $k[X_1, \dots, X_n]$ -modules, k a field, is equivalent to $\text{Serre dim } k[X_1, \dots, X_n] = 0$.

After the solution of Serre's problem by Quillen [16] and Suslin [21], many people worked on surjective stabilization of polynomial extension of a ring. Serre [20] proved $\text{Serre dim } A \leq \dim A$, Plumstead [14] proved $\text{Serre dim } A[X] \leq \dim A$, Bhatwadekar-Roy [4] proved $\text{Serre dim } A[X_1, \dots, X_n] \leq \dim A$ and Bhatwadekar-Lindel-Rao [3] proved $\text{Serre dim } A[X_1, \dots, X_n, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}] \leq \dim A$.

Anderson conjectured an analogue of Quillen-Suslin theorem for monoid algebras over a field which was answered by Gubeladze [8] (see 1.1) as follows.

Theorem 1.1 *Let k be a field and M a monoid. Then M is seminormal if and only if all projective $k[M]$ -modules are free.*

Gubeladze [11] asked the following

Question 1.2 *Let $M \subset \mathbb{Z}_+^r$ be a monoid of rank r with $M \subset \mathbb{Z}_+^r$ an integral extension. Let R be a ring of dimension d . Is $\text{Serre dim } R[M] \leq d$?*

We answer Question 1.2 for some class of monoids. Recall that a finitely generated monoid M of rank r is called Φ -simplicial if M can be embedded in \mathbb{Z}_+^r and the extension $M \subset \mathbb{Z}_+^r$ is integral (see [10]). A Φ -simplicial monoid is commutative, cancellative and torsion-free.

Definition 1.3 Let $\mathcal{C}(\Phi)$ denote the class of seminormal Φ -simplicial monoids $M \subset \mathbb{Z}_+^r$ of rank r such that if $\mathbb{Z}_+^r = \{t_1^{s_1} \dots t_r^{s_r} \mid s_i \geq 0\}$, then for $1 \leq m \leq r$, $M_m = M \cap \{t_1^{s_1} \dots t_m^{s_m} \mid s_i \geq 0\}$ satisfies the following properties: Given a positive integer c , there exist integers $c_i > c$ for $i = 1, \dots, m-1$ such that for any ring R , the automorphism $\eta \in \text{Aut}_{R[t_m]}(R[t_1, \dots, t_m])$ defined by $\eta(t_i) = t_i + t_m^{c_i}$ for $i = 1, \dots, m-1$, restricts to an R -automorphism of $R[M_m]$. It is easy to see that $M_m \in \mathcal{C}(\Phi)$ and rank $M_m = m$ for $1 \leq m \leq r$.

The following result (3.4, 3.8) answers Question 1.2 for monoids in $\mathcal{C}(\Phi)$.

Theorem 1.4 Let $M \subset \mathbb{Z}_+^r$ be a seminormal Φ -simplicial monoid of rank r and R a ring of dimension d .

- (1) If $M \in \mathcal{C}(\Phi)$, then $\text{Serre dim } R[M] \leq d$.
- (2) If $r \leq 3$, then $\text{Serre dim } R[\text{int}(M)] \leq d$, where $\text{int}(M) = \text{int}(\mathbb{R}_+ M) \cap \mathbb{Z}_+^3$ and $\text{int}(\mathbb{R}_+ M)$ is the interior of the cone $\mathbb{R}_+ M \subset \mathbb{R}^3$ with respect to Euclidean topology.

The following result (3.6) follows from (1.4(1)). When R is a field, this result is due to Anderson [1].

Theorem 1.5 Let R be a ring of dimension d and $M \subset \mathbb{Z}_+^2$ a normal monoid of rank 2. Then $\text{Serre dim } R[M] \leq d$.

The next result answers Question 1.2 partially for 1-dimensional rings (see 3.13, 3.16). The proof uses the techniques of Kang [12], Roy [17] and Gubeladze's [9]. Let us recall two definitions. (i) A monoid M is called c -divisible, where $c > 1$ is an integer, if $cX = m$ has a solution in M for all $m \in M$. All c -divisible monoids are seminormal. (ii) Let R be a ring, \overline{R} the integral closure of R and C the conductor ideal of $R \subset \overline{R}$. Then R is called *uni-branched* if for any $\mathfrak{p} \in \text{Spec } R$ containing C , there is a unique $\mathfrak{q} \in \text{Spec } \overline{R}$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.

Theorem 1.6 Let R be a ring of dimension 1, M a monoid and P a projective $R[M]$ -module of rank r .

- (i) If M is c -divisible and $r \geq 3$, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.
- (ii) If R is a uni-branched affine algebra over an algebraically closed field, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.

If R is a 1-dimensional anodal ring with finite seminormalization, then (1.6(ii)) is due to Sarwar ([18], Theorem 1.2). Note that if k is an algebraically closed field of characteristic 2, then node $k[X, Y]/(X^2 - Y^2 - Y^3)$ is not anodal but is uni-branched, by Kang ([12], Example 2).

At the end, we give some applications to minimum number of generators of projective modules.

2 Preliminaries

Let A be a ring and Q an A -module. We say $p \in Q$ is unimodular if the order ideal $O_Q(p) = \{\phi(p) \mid \phi \in \text{Hom}(Q, A)\}$ equals A . The set of all unimodular elements in Q is denoted by $\text{Um}(Q)$. We write $E_n(A)$ for the group generated by set of all $n \times n$ elementary matrices over A and $\text{Um}_n(A)$ for $\text{Um}(A^n)$. We denote by $\text{Aut}_A(Q)$, the group of all A -automorphisms of Q .

For an ideal J of A , we denote by $E(A \oplus Q, J)$, the subgroup of $\text{Aut}_A(A \oplus Q)$ generated by all the automorphisms $\Delta_{a\phi} = \begin{pmatrix} 1 & a\phi \\ 0 & \text{id}_Q \end{pmatrix}$ and $\Gamma_q = \begin{pmatrix} 1 & 0 \\ q & \text{id}_Q \end{pmatrix}$ with $a \in J$, $\phi \in Q^*$ and $q \in Q$. Further, we shall write $E(A \oplus Q)$ for $E(A \oplus Q, A)$. We denote by $\text{Um}(A \oplus Q, J)$ the set of all $(a, q) \in \text{Um}(A \oplus Q)$ with $a \in 1 + J$ and $q \in JQ$.

We state some results of Lindel [13] for later use.

Proposition 2.1 (Lindel [13], 1.1) *Let A be a ring and Q an A -module. Let Q_s be free of rank r for some $s \in A$. Then there exist $p_1, \dots, p_r \in Q$, $\phi_1, \dots, \phi_r \in Q^*$ and $t \geq 1$ such that following holds:*

- (i) $0 :_A s'A = 0 :_A s'^2 A$, where $s' = s^t$.
- (ii) $s'Q \subset F$ and $s'Q^* \subset G$, where $F = \sum_{i=1}^r Ap_i \subset Q$ and $G = \sum_{i=1}^r A\phi_i \subset Q^*$.
- (iii) the matrix $(\phi_i(p_j))_{1 \leq i, j \leq r} = \text{diagonal}(s', \dots, s')$. We say F and G are s' -dual submodules of Q and Q^* respectively.

Proposition 2.2 (Lindel [13], 1.2, 1.3) *Let A be a ring and Q an A -module. Assume Q_s is free of rank r for some $s \in A$. Let F and G be s -dual submodules of Q and Q^* respectively. Then*

- (i) for $p \in Q$, there exists $q \in F$ such that $\text{ht}(O_Q(p + sq)A_s) \geq r$.
- (ii) If Q is projective A -module and $\bar{p} \in \text{Um}(Q/sQ)$, then there exists $q \in F$ such that $\text{ht}(O_Q(p + sq)) \geq r$.

Proposition 2.3 (Lindel [13], 1.6) *Let Q be a module over a positively graded ring $A = \oplus_{i \geq 0} A_i$ and Q_s be free for some $s \in R = A_0$. Let $T \subseteq A$ be a multiplicatively closed set of homogeneous elements. Let $p \in Q$ be such that $p_{T(1+sR)} \in \text{Um}(Q_{T(1+sR)})$ and $s \in \text{rad}(O_Q(p) + A_+)$, where $A_+ = \oplus_{i \geq 1} A_i$. Then there exists $p' \in p + sA_+Q$ such that $p'_T \in \text{Um}(Q_T)$.*

Proposition 2.4 (Lindel [13], 1.8) *Under the assumptions of (2.3), let $p \in Q$ be such that $O_Q(p) + sA_+ = A$ and $A/O_Q(p)$ is an integral extension of $R/(R \cap O_Q(p))$. Then there exists $p' \in \text{Um}(Q)$ with $p' - p \in sA_+Q$.*

The following result is due to Amit Roy ([17], Proposition 3.4).

Proposition 2.5 *Let A, B be two rings with $f : A \rightarrow B$ a ring homomorphism. Let $s \in A$ be non-zero-divisor such that $f(s)$ is a non-zero-divisor in B . Assume that we have the following cartesian square.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A_s & \xrightarrow{f_s} & B_{f(s)} \end{array}$$

Further assume that $\mathrm{SL}_r(B_{f(s)}) = \mathrm{E}_r(B_{f(s)})$ for some $r > 0$. Let P and Q be two projective A -modules of rank r such that (i) $\wedge^r P \cong \wedge^r Q$, (ii) P_s and Q_s are free over A_s , (iii) $P \otimes_A B \cong Q \otimes_A B$ and $Q \otimes_A B$ has a unimodular element. Then $P \cong Q$.

Definition 2.6 (see [10], Section 6) Let R be a ring and M a Φ -simplicial monoid of rank r . Fix an integral extension $M \hookrightarrow \mathbb{Z}_+^r$. Let $\{t_1, \dots, t_r\}$ be a free basis of \mathbb{Z}_+^r . Then M can be thought of as a monoid consisting of monomials in t_1, \dots, t_r .

For $x = t_1^{a_1} \dots t_r^{a_r}$ and $y = t_1^{b_1} \dots t_r^{b_r}$ in \mathbb{Z}_+^r , define x is *lower* than y if $a_i < b_i$ for some i and $a_j = b_j$ for $j > i$. In particular, t_i is lower than t_j if and only if $i < j$.

For $f \in R[M]$, define the *highest member* $H(f)$ of f as am , where $f = am + a_1 m_1 + \dots + a_k m_k$ with $m, m_i \in M$, $a \in R \setminus \{0\}$, $a_i \in R$ and each m_i is strictly lower than m for $1 \leq i \leq k$.

An element $f \in R[\mathbb{Z}_+^r]$ is called *monic* if $H(f) = at_r^s$, where $a \in R$ is a unit and $s > 0$. An element $f \in R[M]$ is said to be *monic* if f is monic in $R[\mathbb{Z}_+^r]$ via the embedding $R[M] \hookrightarrow R[\mathbb{Z}_+^r]$.

Define M_0 to be the submonoid $\{t_1^{s_1} \dots t_{r-1}^{s_{r-1}} \mid s_i \geq 0\} \cap M$ of M . Clearly M_0 is finitely generated as M is finitely generated. Also $M_0 \hookrightarrow \mathbb{Z}_+^{r-1}$ is integral. Hence M_0 is Φ -simplicial. Further, if M is seminormal, then M_0 is seminormal.

Grade $R[M]$ as $R[M] = R[M_0] \oplus A_1 \oplus A_2 \oplus \dots$, where A_i is the $R[M_0]$ -module generated by the monomials $t_1^{s_1} \dots t_{r-1}^{s_{r-1}} t_r^i \in M$. For an ideal I in $R[M]$, define its leading coefficient ideal $\lambda(I)$ as $\{a \in R \mid \exists f \in I \text{ with } H(f) = am \text{ for some } m \in M\}$. ■

Lemma 2.7 ([10], Lemma 6.5) Let R be a ring and $M \subset \mathbb{Z}_+^r$ a Φ -simplicial monoid. If $I \subseteq R[M]$ is an ideal, then $\mathrm{ht}(\lambda(I)) \geq \mathrm{ht}(I)$, where $\lambda(I)$ is defined in (2.6).

3 Main Theorem

This section contains main results stated in the introduction. We also give some examples of monoids in $C(\Phi)$.

3.1 Over $C(\Phi)$ class of monoids

Lemma 3.1 Let R be a ring and $M \subset \mathbb{Z}_+^r$ a monoid in $C(\Phi)$ of rank r . Let $f \in R[M] \subset R[\mathbb{Z}_+^r] = R[t_1, \dots, t_r]$ with $H(f) = ut_1^{s_1} \dots t_r^{s_r}$ for some unit $u \in R$. Then there exist $\eta \in \mathrm{Aut}_R(R[M])$ such that $\eta(f)$ is a monic polynomial in t_r .

Proof Since $M \in C(\Phi)$, we can choose positive integers c_1, \dots, c_{r-1} such that the automorphism $\eta \in \mathrm{Aut}_{R[t_r]} R[t_1, \dots, t_r]$ defined by $\eta(t_i) = t_i + t_r^{c_i}$ for $i = 1, \dots, r-1$, restricts to an automorphism of $R[M]$ and such that $\eta(f)$ is a monic polynomial in t_r . ■

Lemma 3.2 Let R be a ring of dimension d and $M \subset \mathbb{Z}_+^r$ a monoid in $C(\Phi)$ of rank r . Let P be a projective $R[M]$ -module of rank $> d$. Write $R[M] = R[M_0] \oplus A_1 \oplus A_2 \dots$, as defined in (2.6) and

$A_+ = A_1 \oplus A_2 \oplus \dots$ an ideal of $R[M]$. Assume that P_s is free for some $s \in R$ and P/sA_+P has a unimodular element. Then the natural map $\text{Um}(P) \rightarrow \text{Um}(P/sA_+P)$ is surjective. In particular, P has a unimodular element.

Proof Write $A = R[M]$. Since every unimodular element of P/sA_+P can be lifted to a unimodular element of P_{1+sA_+} , if s is nilpotent, then elements of $1+sA_+$ are units in A and we are done. Therefore, assume that s is not nilpotent.

Let $p \in P$ be such that $\bar{p} \in \text{Um}(P/sA_+P)$. Then $O_P(p) + sA_+ = A$. Hence $O_P(p)$ contains an element of $1+sA_+$. Choose $g \in A_+$ such that $1+sg \in O_P(p)$. Applying (2.2) with sg in place of s , we get $q \in F \subset P$ such that $\text{ht}(O_P(p+sgq)) > d$. Since $p+sgq$ is a lift of \bar{p} , replacing p by $p+sgq$, we may assume that $\text{ht}(O_P(p)) > d$. By (2.7), we get $\text{ht}(\lambda(O_P(p))) \geq \text{ht}(O_P(p)) > d$. Since $\lambda(O_P(p))$ is an ideal of R , we get $1 \in \lambda(O_P(p))$. Hence there exists $f \in O_P(p)$ such that the coefficient of $H(f)$ (highest member of f) is a unit.

Suppose $H(f) = ut_1^{s_1} \dots t_r^{s_r}$ with u a unit in R . Since $M \in \mathcal{C}(\Phi)$, by (3.1), there exists $\alpha \in \text{Aut}_R(R[M])$ such that $\alpha(f)$ is monic in t_r . Thus we may assume that $O_P(p)$ contains a monic polynomial in t_r . Hence $A/O_P(p)$ is an integral extension of $R[M_0]/(O_P(p) \cap R[M_0])$ and $\bar{p} \in \text{Um}(P/sA_+P)$. By (2.4), there exists $p' \in \text{Um}(P)$ such that $p' - p \in sA_+P$. This means $p' \in \text{Um}(P)$ is a lift of \bar{p} . This proves the result. \blacksquare

Remark 3.3 In (3.2), we do not need the monoid M to be seminormal.

The next result proves (1.4(1)).

Theorem 3.4 Let R be a ring of dimension d and M a monoid in $\mathcal{C}(\Phi)$ of rank r . If P is a projective $R[M]$ -module of rank $r' \geq d+1$, then P has a unimodular element. In other words, Serre $\dim R[M] \leq d$.

Proof We can assume that the ring is reduced with connected spectrum. If $d = 0$, then R is a field. Since M is seminormal, projective $R[M]$ -modules are free, by (1.1). If $r = 0$, then $M = 0$ and we are done by Serre [20]. Assume $d, r \geq 1$ and use induction on d and r simultaneously.

If S is the set of all non-zero-divisor of R , then $\dim S^{-1}R = 0$ and so $S^{-1}P$ is free $S^{-1}R[M]$ -module ($d = 0$ case). Choose $s \in S$ such that P_s is free. Consider the ring $R[M]/(sR[M]) = (R/sR)[M]$. Since $\dim R/sR = d-1$, by induction on d , $\text{Um}(P/sP)$ is non-empty.

Write $R[M] = R[M_0] \oplus A_1 \oplus A_2 \dots$, as defined in (2.6) and $A_+ = A_1 \oplus A_2 \oplus \dots$ an ideal of $R[M]$. Note that $M_0 \in \mathcal{C}(\Phi)$ and $\text{rank } M_0 = r-1$. Since $R[M]/A_+ = R[M_0]$, by induction on r , $\text{Um}(P/A_+P)$ is non-empty. Write $A = R[M]$ and consider the following fiber product diagram

$$\begin{array}{ccc} A/(sA \cap A_+) & \longrightarrow & A/sA \\ \downarrow & & \downarrow \\ A/A_+ & \longrightarrow & A/(s, A_+) \end{array}$$

If $B = R/sR$, then $A/(s, A_+) = B[M_0]$. Let $u \in \text{Um}(P/A_+P)$ and $v \in \text{Um}(P/sP)$. Let \bar{u} and \bar{v} denote the images of u and v in $P/(s, A_+)P$. Write $P/(s, A_+)P = B[M_0] \oplus P_0$, where P_0 is some projective $B[M_0]$ -module of rank $= r' - 1$. Note that $\dim(B) = d - 1$ and \bar{u}, \bar{v} are two unimodular elements in $B[M_0] \oplus P_0$.

Case 1. Assume $\text{rank}(P_0) \geq \max\{2, d\}$. Then by ([6], Theorem 4.5), there exists $\sigma \in \text{E}(B[M_0] \oplus P_0)$ such that $\sigma(\bar{u}) = \bar{v}$. Lift σ to an element $\sigma_1 \in \text{E}(P/A_+P)$ and write $\sigma_1(u) = u_1 \in \text{Um}(P/A_+P)$. Then images of u_1 and v are same in $P/(s, A_+)P$. Patching u_1 and v over $P/(s, A_+)P$ in the above fiber product diagram, we get an element $p \in \text{Um}(P/(sA \cap A_+)P)$.

Note $sA \cap A_+ = sA_+$. We have P_s is free and P/sA_+P has a unimodular element. Use (3.2), to conclude that P has a unimodular element.

Case 2. Now we consider the remaining case, namely $d = 1$ and $\text{rank}(P) = 2$. Since $B = R/sR$ is 0 dimensional, projective modules over $B[M_0]$ and $B[M]$ are free, by (1.1). In particular, P/sP and $P/(s, A_+)P$ are free modules of rank 2 over the rings $B[M]$ and $B[M_0]$ respectively. Consider the same fiber product diagram as above.

Since any two unimodular elements in $\text{Um}_2(B[M_0])$ are connected by an element of $\text{GL}_2(B[M_0])$. Further $B[M_0]$ is a subring of $B[M] = A/sA$. Hence the natural map $\text{GL}_2(B[M]) \rightarrow \text{GL}_2(B[M_0])$ is surjective. Hence any automorphism of $P/(s, A_+)P$ can be lifted to an automorphism of P/sP . By same argument as above, patching unimodular elements of P/sP and P/A_+P , we get a unimodular element in $P/(sA \cap A_+)P$. Since $sA \cap A_+ = sA_+$ and P/sA_+P has a unimodular element, by (3.2), P has a unimodular element. This completes the proof. ■

Example 3.5 (1) If M is a Φ -simplicial normal monoid of rank 2, then $M \in \mathcal{C}(\Phi)$. To see this, by ([10], Lemma 1.3), $M \cong (\alpha_1, \alpha_2) \cap \mathbb{Z}_+^2$, where $\alpha_1 = (a, b)$ and $\alpha_2 = (0, c)$ and (α_1, α_2) is the group generated by α_1 and α_2 . It is easy to see that $M \cong ((1, a_1), (0, a_2)) \cap \mathbb{Z}_+^2$, where $\gcd(b, c) = g$ and $a_1 = b/g$, $a_2 = c/g$. Hence $M \in \mathcal{C}(\Phi)$.

(2) If $M \subset \mathbb{Z}_+^2$ is a finitely generated rank 2 normal monoid, then it is easy to see that M is Φ -simplicial. Hence $M \in \mathcal{C}(\Phi)$ by (1).

(3) If M is a rank 3 normal quasi-truncated or truncated monoid (see [10], Definition 5.1), then $M \in \mathcal{C}(\Phi)$. To see this, by ([10], Lemma 6.6), M satisfies properties of (1.3). Further, M_0 is a Φ -simplicial normal monoid of rank 2. By (1), $M_0 \in \mathcal{C}(\Phi)$. ■

Corollary 3.6 Let R be a ring of dimension d and $M \subset \mathbb{Z}_+^2$ a normal monoid of rank 2. Then $\text{Serre dim } R[M] \leq d$.

Proof If M is finitely generated, then result follows from (3.5(2)) and (3.4).

If M is not finitely generated, then write M as a filtered union of finitely generated submonoids, say $M = \cup_{\lambda \in I} M_\lambda$. Since M is normal, the integral closure \overline{M}_λ of M_λ is contained in M . Hence $M = \cup_{\lambda \in I} \overline{M}_\lambda$. By ([5], Proposition 2.22), \overline{M}_λ is finitely generated. If P is a projective $R[M]$ -module, then P is defined over $R[\overline{M}_\lambda]$ for some $\lambda \in I$ as P is finitely generated. Now the result follows from (3.5(2)) and (3.4). ■

The following result follows from (3.5(3)) and (3.4).

Corollary 3.7 *Let R be a ring of dimension d and M a truncated or normal quasi-truncated monoid of rank ≤ 3 . Then $\text{Serre dim } R[M] \leq d$.*

Now we prove (1.4(2)).

Proposition 3.8 *Let R be a ring of dimension d and M a Φ -simplicial seminormal monoid of rank ≤ 3 . Then $\text{Serre dim } R[\text{int}(M)] \leq d$.*

Proof Recall that $\text{int}(M) = \text{int}(\mathbb{R}_+ M) \cap \mathbb{Z}_+^3$. Let P be a projective $R[\text{int}(M)]$ -module of rank $\geq d + 1$. Since M is seminormal, by ([5], Proposition 2.40), $\text{int}(M) = \text{int}(\overline{M})$, where \overline{M} is the normalization of M . Since normalization of a finitely generated monoid is finitely generated (see [5], Proposition 2.22), \overline{M} is a Φ -simplicial normal monoid. By ([10], Theorem 3.1), $\text{int}(M) = \text{int}(\overline{M})$ is a filtered union of truncated (normal) monoids (see [10], Definition 2.2). Since P is finitely generated, we get P is defined over $R[N]$, where $N \subset \text{int}(M)$ is a truncated monoid. By (3.7), $\text{Serre dim } R[N] \leq d$. Hence P has a unimodular element. Therefore $\text{Serre dim } R[\text{int}(M)] \leq d$. ■

Assumptions: In the following examples, R is a ring of dimension d , Monoid operations are written multiplicatively and $K(M)$ denotes the group of fractions of monoid M .

Example 3.9 For $n > 0$, consider the monoid $M \subset \mathbb{Z}_+^r$ generated by $\{t_1^{i_1} t_2^{i_2} \dots t_r^{i_r} \mid \sum i_j = n\}$. Then M is a Φ -simplicial normal monoid. For integers $c_i = nk_i + 1$, $k_i > 0$ and $i = 1, \dots, r-1$, consider $\eta \in \text{Aut}_{R[t_r]}(R[t_1, \dots, t_r])$ defined by $t_i \mapsto t_i + t_r^{c_i}$ for $i = 1, \dots, r-1$.

A typical monomial in the expansion of $\eta(t_1^{i_1} \dots t_{r-1}^{i_{r-1}} t_r^{i_r}) = (t_1 + t_r^{c_1})^{i_1} \dots (t_{r-1} + t_r^{c_{r-1}})^{i_{r-1}} t_r^{i_r}$ will look like $(t_1^{i_1 - l_1} t_r^{c_1 l_1}) \dots (t_{r-1}^{i_{r-1} - l_{r-1}} t_r^{c_{r-1} l_{r-1}}) t_r^{i_r} = (t_1^{i_1 - l_1} \dots t_{r-1}^{i_{r-1} - l_{r-1}} t_r^{l_1 + \dots + l_{r-1} + i_r}) t_r^{n(k_1 l_1 + \dots + k_{r-1} l_{r-1})}$ which belong to M . So $\eta(R[M]) \subset R[M]$. Similarly, $\eta^{-1}(R[M]) \subset R[M]$. Hence η restricts to an R -automorphism of $R[M]$. Therefore η satisfies the property of (1.3) for M . It is easy to see that $M_m = M \cap \{t_1^{s_1} \dots t_m^{s_m} \mid s_i \geq 0\}$ for $1 \leq m \leq r-1$ also satisfy this property. Hence $M \in \mathcal{C}(\Phi)$. By (3.4), $\text{Serre dim } R[M] \leq d$. ■

Example 3.10 Let M be a Φ -simplicial monoid generated by monomials $t_1^2, t_2^2, t_3^2, t_1 t_3, t_2 t_3$. For integers $c_j = 2k_j - 1$ with $k_j > 1$, consider the automorphism $\eta \in \text{Aut}_{R[t_3]}(R[t_1, t_2, t_3])$ defined by $t_j \mapsto t_j + t_3^{c_j}$ for $j = 1, 2$. Then it is easy to see that η restricts to an automorphism of $R[M]$.

We claim that M is seminormal but not normal. For this, let

$$z = (t_3^2)^{-1} (t_1 t_3) (t_2 t_3) = t_1 t_2 \in K(M) \setminus M, \text{ but } z^2 \in M,$$

showing that M is not normal. For seminormality, let

$$z = (t_1^2)^{\alpha_1} (t_2^2)^{\alpha_2} (t_3^2)^{\alpha_3} (t_1 t_3)^{\alpha_4} (t_2 t_3)^{\alpha_5} \in K(M) \text{ with } \alpha_i \in \mathbb{Z} \text{ and } z^2, z^3 \in M.$$

We may assume that $0 \leq \alpha_4, \alpha_5 \leq 1$. Now $z^2 \in M \Rightarrow \alpha_1, \alpha_2 \geq 0$ and $2\alpha_3 + \alpha_4 + \alpha_5 \geq 0$. If $\alpha_3 < 0$, then $\alpha_4 = \alpha_5 = 1$ and $\alpha_3 = -1$. In this case, $z^3 = (t_1^{2\alpha_1+1}t_2^{2\alpha_2+1})^3 \notin M$, a contradiction. Therefore $\alpha_3 \geq 0$ and $z \in M$. Hence M is seminormal. It is easy to see that $M \in \mathcal{C}(\Phi)$. By (3.4), $\text{Serre dim } R[t_1^2, t_2^2, t_3^2, t_1t_3, t_2t_3] \leq d$. ■

Remark 3.11 (1) Let R be a ring and P a projective R -module of rank ≥ 2 . Let \overline{R} be the seminormalization of R . It follows from arguments in Bhatwadekar ([2], Lemma 3.1) that $P \otimes_R \overline{R}$ has a unimodular element if and only if P has a unimodular element.

(2) Assume R is a ring of dimension d and $M \in \mathcal{C}(\Phi)$. Let \overline{M} be the seminormalization of M . If \overline{M} is in $\mathcal{C}(\Phi)$, then $\text{Serre dim } R[M] \leq \max\{1, d\}$, using ([2] and 3.4).

(3) Let (R, \mathfrak{m}, K) be a regular local ring of dimension d containing a field k such that either $\text{char } k = 0$ or $\text{char } k = p$ and $\text{tr-deg } K/\mathbb{F}_p \geq 1$. Let M be a seminormal monoid. Then, using Popescu ([15], Theorem 1) and Swan ([23], Theorem 1.2), we get $\text{Serre dim } R[M] = 0$. If M is not seminormal, then $\text{Serre dim } R[M] = 1$ using ([11], [2] and [23]).

Example 3.12 For a monoid M , \overline{M} denotes the seminormalization of M .

1. Let $M \subset \mathbb{Z}_+^2$ be a Φ -simplicial monoid generated by t_1^n, t_1t_2, t_2^n , where $n \in \mathbb{N}$. We claim that M is normal. To see this, let $z = t_1^i t_2^j = (t_1^n)^p (t_1t_2)^q (t_2^n)^r \in K(M)$ with $p, q, r \in \mathbb{Z}$ such that $z^t \in M$ for some $t > 0$. Then $i, j \geq 0$. We need to show that $z \in M$. We may assume that $0 \leq q < n$. Since $i, j \geq 0$, we get $p, r \geq 0$. Thus $z \in M$ and M is normal. Hence, by (3.6), $\text{Serre dim } R[t_1^n, t_1t_2, t_2^n] \leq d$.
2. The monoid $M \subset \mathbb{Z}_+^2$ generated by $t_1^2, t_1t_2^2, t_2^2$ is seminormal but not normal. For this, let $z = (t_1t_2^2)(t_2^2)^{-1} = t_1 \in K(M) \setminus M$. Then $z^2 \in M$ showing that M is not normal. For seminormality, let $z = (t_1^2)^\alpha (t_1t_2^2)^\beta (t_2^2)^\gamma \in K(M)$ with $\alpha, \beta, \gamma \in \mathbb{Z}$ be such that $z^2, z^3 \in M$. We may assume $0 \leq \beta \leq 1$. If $\beta = 0$, then $\alpha, \gamma \geq 0$ and hence $z \in M$. If $\beta = 1$, then $z^2 \in M$ implies $\alpha \geq 0$ and $\gamma + 1 \geq 0$. If $\gamma = -1$, then $z^3 = (t_1)^{6\alpha+3} \notin M$, a contradiction. Hence $\gamma \geq 0$, proving that $z \in M$ and M is seminormal. It is easy to see that $M \in \mathcal{C}(\Phi)$. Therefore, by (3.4), $\text{Serre dim } R[t_1^2, t_1t_2^2, t_2^2] \leq d$.
3. Let M be a monoid generated by $(t_1^2, t_1t_2^j, t_2^2)$, where $j \geq 3$. Then M is not seminormal. For this, if $z = (t_1t_2^j)(t_2^2)^{-1} = t_1t_2^{j-2} \in K(M) \setminus M$, then $z^2 = t_1^2t_2^{2(j-2)}$ and $z^3 = (t_1^2)(t_1t_2^j)(t_2^{2j-6})$ are in M , showing that M is not seminormal.

If $j = 3$, then observe that t_1t_2 belongs to \overline{M} . Since the monoid generated by t_1^2, t_1t_2, t_2^2 is normal, we get that \overline{M} is generated by t_1^2, t_1t_2, t_2^2 . Hence $\text{Serre dim } R[\overline{M}] \leq d$ by (1) above.

Observe that if j is odd, then $\overline{M} = (t_1^2, t_1t_2, t_2^2)$ and if j is even, then $\overline{M} = (t_1^2, t_1t_2^2, t_2^2)$. So $\text{Serre dim } R[\overline{M}] \leq d$ by (1, 2) above.

In both cases, applying (3.11(1)), we get $\text{Serre dim } R[M] \leq \max\{1, d\}$.

4. Let M be a monoid generated by $(t_1^3, t_1 t_2^2, t_2^3)$. Then M is not seminormal. For this, let $z = (t_1 t_2^2)^2 t_2^{-3} \in K(M) \setminus M$. Then $z^2 = t_1^3 (t_1 t_2^2) \in M$ and $z^3 = t_1^6 t_2^3 \in M$. Hence seminormalization of M is $\overline{M} = (t_1^3, t_1^2 t_2, t_1 t_2^2, t_2^3)$. By (3.9), $\text{Serre dim } R[\overline{M}] \leq d$. Therefore, applying (3.11(1)), we get $\text{Serre dim } R[M] \leq \max \{1, d\}$. ■

3.2 Monoid algebras over 1-dimensional rings

The following result proves (1.6(i)).

Theorem 3.13 *Let R be a ring of dimension 1 and M a c -divisible monoid. If P is a projective $R[M]$ -module of rank $r \geq 3$, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.*

Proof If R is normal, then we are done by Swan [23]. Assume R is not normal.

Case 1. Assume R has finite normalization. Let \overline{R} be the normalization of R and C the conductor ideal of the extension $R \subset \overline{R}$. Then $\text{ht } C = 1$. Hence R/C and \overline{R}/C are zero dimensional rings. Consider the following fiber product diagram

$$\begin{array}{ccc} R[M] & \longrightarrow & \overline{R}[M] \\ \downarrow & & \downarrow \\ (R/C)[M] & \longrightarrow & (\overline{R}/C)[M] \end{array}$$

If $P' = \wedge^r P \oplus R[M]^{r-1}$, then by Swan [23], $P \otimes \overline{R}[M] \cong \wedge^r (P \otimes \overline{R}[M]) \oplus \overline{R}[M]^{r-1} \cong P' \otimes \overline{R}[M]$. By Gubeladze [8], P/CP and P'/CP' are free $(R/C)[M]$ -modules. Further, $\text{SL}_r((\overline{R}/C)[M]) = \text{E}_r((\overline{R}/C)[M])$ for $r \geq 3$, by Gubeladze [9]. Now using standard arguments of fiber product diagram, we get $P \cong P'$.

Case 2. Now R need not have finite normalization. We may assume R is a reduced ring with connected spectrum. Let S be the set of all non-zero-divisors of R . By [8], $S^{-1}P$ is a free $S^{-1}R[M]$ -module. Choose $s \in S$ such that P_s is a free $R_s[M]$ -module.

Now we follow the arguments of Roy ([17], Theorem 4.1). Let \hat{R} denote the s -adic completion R . Then \hat{R}_{red} has a finite normalization. Consider the following fiber product diagram

$$\begin{array}{ccc} R[M] & \longrightarrow & \hat{R}[M] \\ \downarrow & & \downarrow \\ R_s[M] & \longrightarrow & \hat{R}_s[M] \end{array}$$

Since \hat{R}_s is a zero dimensional ring, by [9], $\text{SL}_r(\hat{R}_s[M]) = \text{E}_r(\hat{R}_s[M])$ for $r \geq 3$. If $P' = \wedge^r P \oplus R[M]^{r-1}$, then P_s and P'_s are free $R_s[M]$ -modules and by Case 1, $P \otimes \hat{R}[M] \cong P' \otimes \hat{R}[M]$. By (2.5), $P \cong P'$. This completes the proof. ■

The following result is due to Kang ([12], Lemma 7.1 and Remark).

Lemma 3.14 *Let R be a 1-dimensional uni-branched affine algebra over an algebraically closed field, \overline{R} the normalization of R and C the conductor ideal of the extension $R \subset \overline{R}$. Then $\overline{R}/C = R/C + a_1R/C + \cdots + a_mR/C$, where $a_i \in \sqrt{C}$ the radical ideal of C in \overline{R} .*

Lemma 3.15 *Let R be a 1-dimensional ring, \overline{R} the normalization of R and C the conductor ideal of the extension $R \subset \overline{R}$. Assume $\overline{R}/C = R/C + a_1R/C + \cdots + a_mR/C$, where $a_i \in \sqrt{C}$ the radical ideal of C in \overline{R} . Let M be a monoid and write $A = \overline{R}/C$.*

- (i) *If $\sigma \in \mathrm{SL}_n(A[M])$, then $\sigma = \sigma_1\sigma_2$, where $\sigma_1 \in \mathrm{SL}_n((R/C)[M])$ and $\sigma_2 \in \mathrm{E}_n(A[M])$.*
- (ii) *If P is a projective $R[M]$ -module of rank r , then $P \cong \wedge^r P \oplus R[M]^{r-1}$.*

Proof (i) Let $\sigma = (b_{ij}) \in \mathrm{SL}_n(A[M])$. Write $b_{ij} = (b_{ij})_0 + (b_{ij})_1a_1 + \cdots + (b_{ij})_ma_m$, where $(b_{ij})_l \in (R/C)[M]$. If $\alpha = ((b_{ij})_0)$, then $\det(\sigma) = 1 = \det(\alpha) + c$, where $c \in (\sqrt{C}/C)[M]$. Since $c \in (R/C)[M]$ is nilpotent, $\det(\alpha)$ is a unit in $(R/C)[M]$. Let $\beta = \text{diagonal}(1/(1-c), 1, \dots, 1) \in \mathrm{GL}_n((R/C)[M])$ and $\sigma_1 = \alpha\beta \in \mathrm{SL}_n((R/C)[M])$.

Note that $\sigma_1^{-1}\sigma = \beta^{-1}\alpha^{-1}\sigma = \beta^{-1}1/(1-c)\overline{\alpha}\sigma$, where $\overline{\alpha} = ((\overline{b}_{ij})_0)$, $(\overline{b}_{ij})_0$ are minors of $(b_{ij})_0$.

$$\sigma_2 := \sigma_1^{-1}\sigma = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{1-c} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{1-c} \end{bmatrix} \begin{bmatrix} 1+c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & 1+c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1} & c_{n2} & \cdots & 1+c_{nn} \end{bmatrix},$$

where $c_{ij} \in (\sqrt{C}/C)[M]$.

Note that $\sigma_2 \in \mathrm{SL}_n(A[M])$ and $\sigma_2 = Id$ modulo the nilpotent ideal of $A[M]$. Hence $\sigma_2 \in \mathrm{E}_n(A[M])$. Thus we get $\sigma = \sigma_1\sigma_2$ with the desired properties.

- (ii) Follow the proof of (3.13) and use (3.15(i)) to get the result. ■

Now we prove (1.6(ii)) which follows from (3.14) and (3.15).

Theorem 3.16 *Let R be a 1-dimensional uni-branched affine algebra over an algebraically closed field and M a monoid. If P is a projective $R[M]$ -module of rank r , then $P \cong \wedge^r P \oplus R[M]^{r-1}$.*

4 Applications

Let R be a ring of dimension d and Q a finitely generated R -module. Let $\mu(Q)$ denote the minimum number of generators of Q . By Forster [7] and Swan [22], $\mu(Q) \leq \max\{\mu(Q_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(R), Q_{\mathfrak{p}} \neq 0\}$. In particular, if P is a projective R -module of rank r , then $\mu(P) \leq r + d$.

The following result is well known.

Theorem 4.1 *Let A be a ring such that $\mathrm{Serre} \dim A \leq d$. Assume A^m is cancellative for $m \geq d+1$. If P is a projective A -module of rank $r \geq d+1$, then $\mu(P) \leq r + d$.*

Proof Assume $\mu(P) = n > r + d$. Consider a surjection $\phi : A^n \twoheadrightarrow P$ with $Q = \ker(\phi)$. Then $A^n \cong P \oplus Q$. Since Q is a projective A -module of rank $\geq d + 1$, Q has a unimodular element q . Since $\phi(q) = 0$, ϕ induces a surjection $\bar{\phi} : A^n/qA^n \twoheadrightarrow P$. Since $n - 1 > d$, A^{n-1} is cancellative. Hence $A^{n-1} \cong A^n/qA$ and P is generated by $n - 1$ elements, a contradiction. ■

The following result is immediate from (4.1, 3.4, 3.6 and [6]).

Corollary 4.2 *Let R be a ring of dimension d , M a monoid and P a projective $R[M]$ -module of rank $r > d$. Then:*

- (i) *If $M \in \mathcal{C}(\Phi)$, then $\mu(P) \leq r + d$.*
- (ii) *If $M \subset \mathbb{Z}_+^2$ is a normal monoid of rank 2, then $\mu(P) \leq r + d$.*

Let M be a c -divisible monoid, R a ring of dimension d and $n \geq \max\{2, d + 1\}$. Then Schaubhüser [19] proved that $E_{n+1}(R[M])$ acts transitively on $\text{Um}_{n+1}(R[M])$. Using Schaubhüser's result and arguments of Dhorajia-Keshari ([6], Theorem 4.4), we get that if P is a projective $R[M]$ -module of rank n , then $E(R[M] \oplus P)$ acts transitively on $\text{Um}(R[M] \oplus P)$. Therefore the following result is immediate from (4.1 and 3.13).

Corollary 4.3 *Let R be a ring of dimension 1, M a c -divisible monoid and P a projective $R[M]$ -module of rank $r \geq 3$. Then $\mu(P) \leq r + 1$.*

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